Kinetic model for multidimensional opinion formation

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In this paper, we deal with a kinetic model to describe the evolution of the opinion in a closed group with respect to a choice between multiple options (e.g., political parties), which takes into account two main mechanisms of opinion formation, namely, the interaction between individuals and the effect of the mass media. We numerically test the model in some relevant cases and eventually provide an existence and a uniqueness result for it.

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I. INTRODUCTION

The idea of modeling sociological behaviors by using tools of statistical mechanics arose about thirty years ago. The topics covered by such a research field, called *sociophysics* by Galam *et al.* in the early '80s [1], deal with several different problems, including social networks, population dynamics, voting, coalition formation, and opinion dynamics.

The kinetic theory has only been recently applied to describe collective sociological behaviors (see, for instance [2–4]). During the last few years, the opinion formation with respect to a binary question (typically, a referendum), or to situations requiring a monodimensional opinion variable, has been modeled by kinetic-type equations in [5–9]. The main advantage of the kinetic formulation with respect to other strategies such as, for instance, the Ising models [10–12], is that one can also take into account intermediate opinions, therefore allowing to describe a partial agreement (or disagreement). It is worth noticing that the kinetic description has mainly been employed, up to now, in that case of binary questions.

Overtaking the situations concerning this kind of opinion formation is not completely straightforward from a modeling viewpoint. Indeed, a major problem consists in the fact that, with a plurality of possible options, in general, it is not possible to rank the options independently on the individual. For example, the schematization left/right in politics is not univocal, and some persons, although having a clear political orientation, can vote for a party which has the opposite orientation but defend, some tangible interest that is very important to them.

With the order between parties being a personal matter, it is hence not possible to use only one scalar independent opinion variable, at least when there are more than two political parties. It then becomes necessary to introduce an opinion vector, whose dimension coincides with the number of the possible choices. We note moreover that the opinion on an option can be independent of the opinion on the others ones: the use of a multidimensional opinion variable allows us to take into account situations where the individual has a positive viewpoint for more than one possibility of choice.

In this paper, we give specific attention to a particular problem of multidimensional opinion formation. We aim to tackle the modeling of a large population which must choose between two or more available political parties through a vote. However, our results are not limited to this context. Our strategy can indeed be used in many other situations, for instance, the choice between some products in a nonmonopolistic commercial market.

In a large-scale election, it is well accepted that two different processes play a fundamental role in the phenomenon of opinion formation inside a population: the binary exchange of ideas between individuals and the influence of the mass media (TV networks, radios, newspapers, internet, etc.).

Whereas the interpersonal communication is always an essential ingredient in the time evolution of the public opinion, the interaction with media is typical of some kind of choices. For example, in a small-scale election (such as a local election in a small community), the effect of the media, if any, is often less efficient than the interpersonal exchange of ideas between individuals: here, the direct knowledge of the candidates turns out to be decisive. Note that, in such a context, it would also be convenient to discuss the relevance of the kinetic description since the basic assumption on the large size of the population cannot hold anymore.

This last phenomenon has been very well understood by political actors, who make great effort in order to take advantage from it (e.g., by using direct advertisements but also by controlling directly or indirectly the media themselves).

In this work, we have chosen to model the interpersonal communication by a unique mechanism, which is very similar to the process defined in [6]: the binary exchange between individuals induces concentration toward the majority opinion.

In general, this assumption does not hold. It is indeed clear that, in a real situation, the phenomenology is much more intricate. For example, many different behaviors of opinion formation depend on the fact that the way people think is not uniform. A realistic model should therefore include as many binary interaction rules between individuals as the mental paths of the members of the population. It is worth noticing that the variety of behaviors in the context of interpersonal communication already allows explanation of many interesting phenomena, such as the concentration toward some particular opinions or the cyclic (in time) behavior of the distribution function: we refer to [9,13] in the case of models depending on one scalar variable.

However, one of our primary objectives is to focus on the influence of the mass media on the population. Since the introduction of other factors which can modify the opinion evolution can hide the direct impact of the mass media, we shall therefore disregard them and only consider a unique interaction rule coupled with the media effect. In this context, the binary collision rule which tends to the consensus seems the best choice since it represents a very popular way of thinking and, in many populations, is the most common behavior [14–17]. The next step in the modeling analysis should be to consider a more complete model, with a plurality of possible effects as the ones briefly mentioned above.

Concerning the characteristics of the media, we choose to consider the media opinion as an external input to the population. It means that someone, who is not influenced by the population itself, tries to modify the equilibrium which would be reached in a media-free situation.

Our choice for the media characteristics has the advantage of permitting the study of possible manipulation effects on the population. This kind of situations has been individuated and analyzed [18,19]. There exist, indeed, special interest groups which are able to manipulate the public opinion through the media, both in democratic societies and in autocratic ones.

Once the aforementioned phenomena are established, the dynamics of their competition is clear. Whereas the binary exchange between individuals induces concentration toward a weighted majority opinion, the presence of the media disturbs this tendency to compromise through an attraction effect toward the media opinion.

We point out that our model describes the evolution of the opinion in a community with respect to a multidimensional choice, but it does not provide any forecast on the choice itself. By analogy with quantum mechanics, we might say that our model foresees the time evolution of the state of a system, whereas the choice process is the analogous of the measurement process, which gives, as a result, an eigenstate of the system. In an electoral process, when there are more than two options, the electoral system plays a crucial role, and the translation of the voter's sympathy into a vote heavily depends on the voting process. If the system is purely proportional, the voting strategy can be based on a maximal agreement rule. On the other hand, if the individuals are confronted with a majority system (or a mixed one), sometimes they can vote for a party which is not the best one in their own opinion but has real possibilities of winning the election rather than for a party which fits their opinion but is in no position of winning.

Here we choose to only consider a purely proportional voting system, which allows to establish a clear link between the opinion of an individual and its vote. Other systems of vote imply the coupling of our model of opinion formation with a strategy of vote based on game theory [2].

The paper is organized as follows. In Sec. II, we describe our model, written in a weak form with respect to the opinion vector. Note that the number of available political parties is arbitrary, but it is obviously finite. We only performed numerical simulations for two and three parties, and the results are collected in Sec. III. This choice is due to the fact that the distribution function can be visualized only when the available options are at most two. With more than two options, only quantities related to the distribution function (but not the distribution function itself) can be visualized. Moreover, the dynamical complexity of having more than two options is fully present in the three-dimensional case. Eventually, in the last section, we offer an alternate formulation of our problem which helps obtain an existence and a uniqueness result. Of course, this result really matters since it ensures that numerical solutions are not spurious and that the model is mathematically well posed.

II. MODEL FOR OPINION FORMATION

Let us consider an election process with $p \geq 1$ political parties, denoted as P_i, $1 \leq i \leq p$. For each party P_i, we introduce an *agreement variable* $x_i \in [-1,1]$. In the following, Ω denotes the open interval $(-1,1)$.

We label with $x_i=-1$ and $x_i=1$ the two extreme behaviors: the complete disagreement with the party P_i is translated into the model by setting $x_i=-1$, and the opposite situation, i.e., the complete agreement, is translated by setting $x_i=1$. Note that any intermediate value between the two extremes, $x_i=0$ excluded, means partial agreement or disagreement, with a degree of conviction proportional to $|x_i|$. The value $x_i=0$ means total indifference with respect to party P_i.

Since there are several parties, it can be useful to define the *opinion* (or *agreement*) *vector* $x=(x_1,\ldots,x_p) \in \bar{\Omega}^p$, which gives, for each individual of the population, its feelings about the political parties.

The unknown of our model is a density (or distribution function) $f=f(t,x) \geq 0$, defined on $\mathbb{R}_+ \times \bar{\Omega}^p$, whose time evolution is described by a kinetic-type equation. If the agreement vector is defined on a subdomain $D \subseteq \bar{\Omega}^p$, the integral

$$\int_D f(t,x)dx$$

represents the number of individuals with opinion included in D at time $t \geq 0$. Note that, in order to give a meaning to the previous considerations, f should satisfy $f(t,\cdot) \in L^1(\Omega^p)$ for all $t \in \mathbb{R}_+$.

As sketched in Sec. I, we only take into account two processes of opinion evolution. The first one is given by the binary interaction between individuals, who exchange their points of view and adjust their opinions on the ground of each other's belief. The second one is the interaction with the media. Both phenomena are accurately presented below.

A. Exchange of opinions inside the population

We model this process by borrowing the collisional mechanism of a typical interaction in the kinetic theory of gases: whereas, in rarefied gas dynamics, the particles exchange momentum and energy in such a way that the principles of classical mechanics are satisfied, here the "collision" between individuals allows the exchange of opinions.

Let x, $x^* \in \bar{\Omega}^p$ the opinion vectors of two individuals before an interaction. We suppose that the opinions after the interaction change according to the following rule:

$$\begin{cases} x_i' = \dfrac{x_i + x_i^*}{2} + \eta(x_i)\dfrac{x_i - x_i^*}{2}, \\ (x_i^*)' = \dfrac{x_i + x_i^*}{2} + \eta(x_i^*)\dfrac{x_i^* - x_i}{2}, \end{cases} \qquad 1 \le i \le p. \tag{1}$$

Of course, other choices, based on sociological considerations, are possible. For $p=1$, with Eq. (1), we recover the collision rule defined in [6].

Function $\eta:\bar{\Omega}\to\mathbb{R}$, which we henceforth name the function, is smooth and it describes the degree of attraction of the average opinion with respect to the starting opinion of the agent. Note that η may depend on i, but we choose to disregard this dependence, since we obtain the same kind of results. In the sequel, we need some more assumptions on the attraction coefficient η.

Definition 2.1. *Let $\eta:\bar{\Omega}\to\mathbb{R}$ be an even function of class $C^1(\bar{\Omega})$. The attraction function is admissible if:*

(i) $0 \le \eta(s) < 1$ *for all* $s \in \bar{\Omega}$;

(ii) $\eta'(s) < 0$ *for all* $s \in [-1,0]$; *and*

(iii) *the Jacobian $J(x_i,x_i^*)$ of collision mechanism (1), taken component by component; i.e.,*

$$J(x_i,x_i^*) = \frac{1}{2}[\eta(x_i) + \eta(x_i^*)] - \frac{1}{4}\eta'(x_i)\eta'(x_i^*)(x_i - x_i^*)^2 + \frac{1}{4}[\eta'(x_i) - \eta'(x_i^*)](x_i - x_i^*) + \frac{1}{4}[\eta'(x_i)\eta(x_i^*) - \eta(x_i)\eta'(x_i^*)](x_i - x_i^*)$$

is uniformly lower bounded by a strictly positive constant. That is, there exists $J_{\min} > 0$ such that $J(x_i,x_i^*) \ge J_{\min}$ for any i and any couple $(x_i,x_i^*) \in \bar{\Omega}^2$.

The first property prevents that the interaction destroys the bounds of the interval Ω. The second one translates the assumption that the effects of the interaction between individuals are stronger when the precollisional opinions are close to zero. The third one ensures that the inverse of collision rule (1) is well defined.

Remark 2.2. *By using the properties listed in definition 2.1, it is not difficult to also prove that, for any i and x_i, $x_i^* \in \bar{\Omega}$,*

$$x_i' - (x_i^*)' = \frac{1}{2}[\eta(x_i) + \eta(x_i^*)](x_i - x_i^*),$$

and, since $0 \le \eta < 1$,

$$|x_i' - (x_i^*)'| \le |x_i - x_i^*|.$$

It is then clear that the lateral bounds are not violated; i.e.,

$$\max\{|x_i'|,|(x_i^*)'|\} \le \max\{|x_i|,|x_i^*|\}.$$

We note that the set of admissible attraction coefficients is not empty. A possible choice is $\eta(s)=\lambda(1+s^2)$, with $0<\lambda<1/2$.

Once collision rule (1) is defined, the interaction between individuals and the corresponding exchange of opinions is described by a collisional integral of Boltzmann type.

The collisional integral, which is denoted as Q, has the classical structure of the dissipative Boltzmann kernels. At a formal level, it can be viewed as composed of two parts: a *gain term* Q^+, which quantifies the exchanges of opinion between individuals which give, after the interaction with another individual, the opinion vector x, and a *loss term* Q^-, which quantifies the exchanges of opinions where an individual with precollisional opinion vector x experiences an interaction with another member of the population.

It is apparent that the existence of a precollisional pair which restitutes the postcollisional pair (x,x_*) through a collision of type [Eq. (1)] is not guaranteed unless we suppose that the collisional rule is a diffeomorphism of $\bar{\Omega}^{2p}$ onto itself. Unfortunately, collisional mechanism (1) does not verify this property. For instance, in general, there is no $(x_i,x_i^*) \in \bar{\Omega}^2$ which gives, after collision, the couple of extreme opinions $(-1,1)$.

In order to overcome this difficulty, the natural framework for such a collision rule is given by the weak form. Two choices are possible. We may either build a model in a weak form with respect to x only or work in a weak setting with respect to the whole set of independent variables. We choose the first option, which seems to be the correct framework for such kind of models.

A crucial term of the collision integral is given by the cross section. This quantity measures the probability of interaction between individuals and, moreover, the probability that the interaction causes a modification of the agent's opinion. We suppose that the cross section $\beta:\Omega\times\Omega\to\mathbb{R}_+$ is a function of class $L^\infty(\Omega\times\Omega)$ which depends on a suitable precollisional opinion distance.

Let $\varphi=\varphi(x)$ be a regular test function. We define the weak form of the collision kernel as

$$\langle Q(f,f),\varphi\rangle = \int\int_{\Omega^{2p}} \beta(x,x^*)f(t,x)f(t,x^*)[\varphi(x') - \varphi(x)]dx^*dx. \tag{2}$$

Note that the particular form of collision rule (1) only enters through the test function $\varphi(x')$. It is also clear that the operator Q only acts on the agreement vector and not on the time variable.

The explicit form of the change of variables [Eq. (1)] also allows us to give the following alternative formulations of the collision kernel:

$$\langle Q(f,f),\varphi\rangle = \int\int_{\Omega^{2p}} \beta(x,x^*)f(t,x)f(t,x^*)[\varphi((x^*)') - \varphi(x^*)]dx^*dx = \frac{1}{2}\int\int_{\Omega^{2p}} \beta(x,x^*)f(t,x)f(t,x^*) \times[\varphi(x') + \varphi((x^*)') - \varphi(x) - \varphi(x^*)]dx^*dx.$$

Remark 2.3. *At least formally, we have* $\langle Q(f,f),1\rangle = 0$.

B. Media influence

The effects of the media on the population are here modeled by a fixed background. This assumption adds a linear kinetic term into our equations. We consider a set of $m \in \mathbb{N}^*$ media. For any media M_j, $1 \le j \le m$, we introduce two quantities: its *strength* α_j, which translates the influence of the media on the population and its opinion vector $X^j \in \Omega^p$, with respect to each political party.

Both quantities can be time dependent. In what follows, we suppose that the strength of the media is constant. This is the simplest assumption. It seems reasonable if the time scale is small enough, as it may happen during an electoral process. Of course, other choices, based on sociological considerations, are possible.

We do not suppose, however, that the opinion vector of the media is time independent: even if, normally, the opinionists are quite stable in their convictions, some events can considerably modify the appealing of a party. Moreover, a particular strategy of manipulation of the public opinion, which is investigated in Sec. III A, is based on a time evolution of the opinion vector of the media. Hence, in the following, we admit that $X^j:\mathbb{R}_+ \to \Omega^p$ for any j.

The effect of each media M_j on the individual is therefore described by an interaction rule which reminds collision rule (1), that is

$$\tilde{x}_i = \Phi_i^j(x_i) := x_i + \xi_j(|X_i^j - x_i|)(X_i^j - x_i), \tag{3}$$

for all i and j.

The functions $\xi_j:[0,2]\to\mathbb{R}$ are the *influence functions* and satisfy the prescription collected in the following definition:

Definition 2.4. *Let* $1 \le j \le m$ *and* $\xi_j:[0,2]\to\mathbb{R}_+$ *be a function of class* $C^1([0,2])$. *The influence function is admissible if* $0 \le \xi_j(s) < 1$ *and if there exists* $c_j \in (0,2)$ *such that:*

(i) $\xi_j(s)=0$ *for all* $s \in [c_j,2]$;

(ii) $\xi_j'(s) < 0$ *for all* $s \in (0,c_j)$.

Using this definition, we have the following proposition, whose proof is immediate.

Proposition 2.5. *Rule (3) is invertible. More precisely, the function* $\Phi_i^j: x_i \mapsto \tilde{x}_i$ *is a* C^1 *diffeomorphism on* $\bar{\Omega}$ *for any* $j = 1,\ldots,m$ *and for any* $i=1,\ldots,p$.

The set of admissible influence functions is not empty. Indeed, a possible choice of ξ_j, with $c_j=1$ and $0<\lambda<1/2$, is

$$\xi_j(s) = \begin{cases} \lambda(1+\cos(\pi s)) & \text{if } |s| \in [0,1] \\ 0 & \text{otherwise} \end{cases}.$$

The influence function acts in a different manner from the attraction function since it depends on the distance between the opinion of the agent and the opinion of the media. When $\xi_j = 0$, the media has no effect in changing the opinion of the corresponding individual. This hypothesis translates the idea that the media may more easily influence people with a similar opinion.

Once interaction rule (3) is defined, the influence of each media is described by a (possibly time-dependent) linear integral operator, L_j, $1 \le j \le m$, that has the classical structure of the linear Boltzmann kernels. The natural framework is also the weak formulation.

Let $\varphi = \varphi(x)$ be a suitably regular test function. We define the weak form of the interaction kernel as

$$\langle L_j f,\varphi\rangle = \alpha_j \int_{\Omega^p} f(t,x)[\varphi(\tilde{x}) - \varphi(x)]dx. \tag{4}$$

Remark 2.6. *At least formally, we have* $\langle L_j f,1\rangle = 0$.

C. Combining the two phenomena

We are then able to write down the whole model. Let $T > 0$. The evolution law of the unknown $f=f(t,x)$ results in an integrodifferential equation,

$$\int_{\Omega^p} f_t(t,x)\varphi(x)dx = \sum_{j=1}^{m} \langle L_j f,\varphi\rangle + \langle Q(f,f),\varphi\rangle \tag{5}$$

posed in $(t,x) \in [0,T]\times\Omega^p$, for all $\varphi \in C(\Omega^p)$, with initial condition

$$f(0,x) = f^{\text{in}}(x) \text{ for all } x \in \Omega^p. \tag{6}$$

D. Mass conservation

Our model guarantees the conservation of the total number of individuals of the population. By borrowing the kinetic theory language, the following result is also named the total mass conservation.

Proposition 2.7. *Let* $f=f(t,x)$ *be a nonnegative solution of Eqs. (5) and (6), with a nonnegative initial condition* $f^{\text{in}} \in L^1(\Omega^p)$. *Then we have*

$$\|f(t,\cdot)\|_{L^1(\Omega^p)} = \|f^{\text{in}}\|_{L^1(\Omega^p)}, \quad \text{for a.e. } t \ge 0.$$

Proof. We simply consider Eq. (5) with test function $\varphi \equiv 1$. ■

Since $|x| \le 1$, from the mass conservation, we immediately deduce that all the moments of f are bounded.

Of course, the mass conservation is not realistic if we consider long-time forecasts. Indeed, in such situations, we should also consider processes of birth, death, and shift in age of the voters, which would lead to the variation in the total number of individuals. But usually, as in the case of elections or referendums, the interest of such models is to deduce short-term forecasts by using, as an initial datum, the result of some opinion poll.

We must emphasize that the total mass conservation is one of the key properties which must be preserved in the numerical scheme.

III. NUMERICAL SIMULATIONS

This section is devoted to the investigation of the numerical behavior of the model. We limit ourselves to the two-dimensional (2D) and three-dimensional situations, mostly for computational cost and readability reasons. We apply the model to an electoral competition but, as explained in the introduction, other situations with an analogous dynamics (e.g., the choice between some products advertised by media) can be described by the same tool.

The computations are performed using a numerical code written in *C*. We consider a regular subdivision $(x^0,\dots,x^N)$ of $\overline{\Omega}$, with $N\geq 1$. The function f is computed at the center of each cubic cell $\Pi_{i=1}^{p}(x^{k_i},x^{k_i+1})$, $0\leq k_i\leq N-1$. In our computations, we choose $N=100$.

As required in proposition 2.7, the associated numerical scheme conserves the total agents number, i.e., $\|f(t)\|_{L^1_x}$. In order to simulate collisions, we used a slightly modified Bird method [20]. Note that our scheme does not allow the scalar opinions to leave $[-1,1]$. As a matter of fact, opinions x such that $|x|>1$ are not possible because the collision mechanism prevents them, and the media opinions of the media also belong to $[-1,1]$.

In the whole section, the form of the attraction function is $\eta(s)=0.25(1+s^2)$. The influence function (independent of the media) is $\xi(s)=0.9(1-s^2)$ on $[0,1]$ and $\xi(s)=0$ on [1,2]. Note that, of course, this function ξ is not C^1, but that does not matter for the numerics. We also choose a constant cross section $\beta=1$. The values of (α_j) are given as proportional to β. The media features are given for each test.

A. Two-party system

We first choose $p=2$; i.e., the phenomenon of opinion formation only concerns two political parties. Using a scalar opinion variable would then be an option, but, in this case, the meaning of the scalar variable would be the signed difference of the two components of the two-dimensional opinion vector. In fact, the 2D model contains more information about the population opinion than the one-dimensional one. Anyway, we are also interested in the following integrals, which represent the population percentage, respectively, in favor of parties P_1 and P_2:

$$I_1=\int\int_{E_1} f(x)dx,\quad I_2=\int\int_{E_2} f(x)dx=1-I_1,$$

where

$$E_1=\{(x_1,x_2)\in\Omega^2|x_1>x_2\},\quad E_2=\{(x_1,x_2)\in\Omega^2|x_1<x_2\}.$$

When not specified, the population is uniformly distributed: $f^{\text{in}}(x_1,x_2)=0.25$.

1. Population away from the media influence

We first consider the effect of one media whose opinion is too far from the opinion of the population. The initial datum is not uniform; more precisely, we set $f^{\text{in}}(x_1,x_2)=4$ when $(x_1<-0.5,\ x_2<-0.5)$, and 0 otherwise. The opinion of the media is centered in (0.9,0.9), and its strength is $\alpha_1=0.1\beta$.

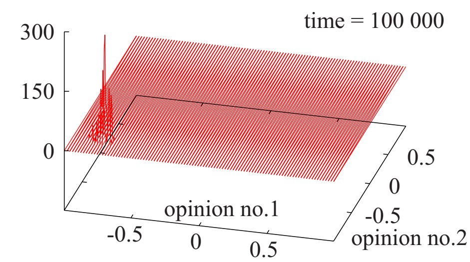

FIG. 1. (Color online) Distribution function at time $t=100\ 000$.

We observe no effect of the media opinion on the population. The distribution function converges toward the Dirac mass centered in $(-0.75,-0.75)$. Figure 1 shows the plot of f at time $t=100\ 000$.

2. Influence of one unique media

We here consider one unique media which can act on the population. When the media opinion is in agreement with the opinion of a part, even very small, of the population, its effect is far from being negligible. The behavior of the model is indeed obtained through the combined effect of compromise effect and influence of the media, which acts as a linear Boltzmann kernel.

Heuristically, the dynamics is the following. When the members of the group with precollisional opinions close to the media opinion interact with the remainder of the population, the opinions of the latter are drawn up toward the media opinion. In this case, a successive interaction with the media can have a significant effect, and the convergence to the media opinion becomes possible.

The model exhibits a threshold effect: if the fraction of the population whose interaction with the media is significant is above a critical value, then we can numerically recover that, asymptotically, the distribution function goes to a Dirac mass centered at the same point of the opinion of the media. Otherwise, the concentration effect of the media is not enough to draw the whole population to the media opinion.

We recover both behaviors in the next two numerical simulations. The media strength is set to $\alpha_1=0.1\beta$, and its opinion vector is (0.9,0.9).

Situation 1. We consider an initial datum such that $f^{\text{in}}(x_1,x_2)=56/15$ when $(x_1<-0.5,\ x_2<-0.5)$, $f^{\text{in}}(x_1,x_2)=8/15$ when $(0.75>x_1>0.5,\ x_2<-0.5)$ and zero otherwise. In Fig. 2, we observe that the concentration effect is

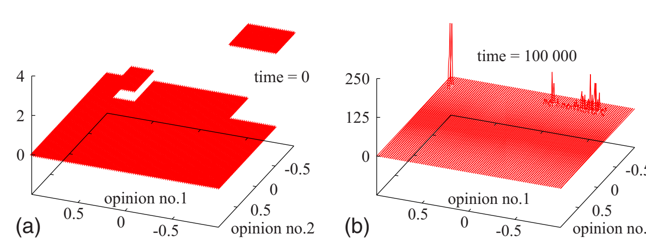

FIG. 2. (Color online) Distribution function for situation 1 at (a) $t=0$ and (b) $t=100\ 000$.

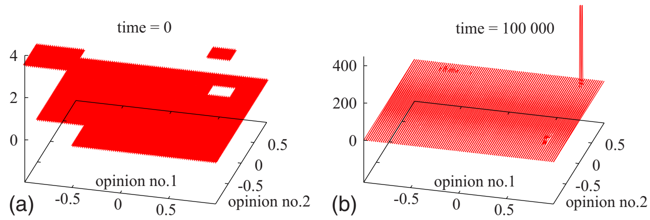

FIG. 3. (Color online) Distribution function for situation 2 at (a) $t=0$ and (b) $t=100\,000$.

not global: the distribution function has a Dirac-like behavior in $(0.9,-0.75)$, but it cannot vanish around the region $(x_1 < -0.1,\ x_2 \simeq -0.75)$ because of the threshold on the media effect.

Situation 2. In this case, the initial datum satisfies: $f^{\text{in}}(x_1,x_2)=32/9$ when $(x_1<-0.5,\ x_2<-0.5)$, $f^{\text{in}}(x_1,x_2)=16/9$ when $(0.75>x_1>0.5,\ 0.75>x_2>0.5)$ and zero otherwise. This time, an almost full concentration effect around the media opinion is shown in Fig. 3.

3. Competition between two fixed unbalanced media

From now on, we only use a population with a uniformly distributed opinion. We study the effect of two media with different strengths. More precisely, we have $\alpha_1=0.1\beta$, $X^1=(0.6,-0.4)$, $\alpha_2=0.3\beta$, and $X^2=(-0.3,0.7)$. In Fig. 4, we observe the forming of two Dirac masseslike in X^2 and in $(-0.15,0.7)$, and the vanishing of two other Dirac-like masses. The highest one is centered on the stronger media opinion. The remaining other one does not vanish when time grows, and its mass is one-third of the one centered in X^2.

4. Opinion manipulation by a media

We choose two media with the same strength $\alpha_1=\alpha_2=0.1\beta$, and the opinion vector of media M_1 is fixed $X^1=(0.4,-0.4)$.

In Fig. 5, up to time 10 000, we compare the behavior of I_1 in the two following cases. We first choose $X^2=(-0.4,0.4)$, constant with respect to time. Then we choose X^2 with the same opinion, except when $3000<t<7000$, where $X^2=(-0.39,0.39)$.

One can check that, with these two choices, the distribution function centers on (0,0) when time grows but does not become a Dirac mass. However, the impact of a variable

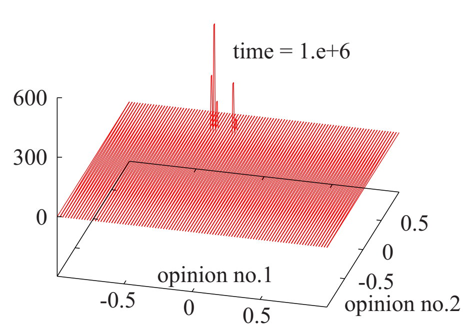

FIG. 4. (Color online) Distribution function with two media at $t=1\,000\,000$.

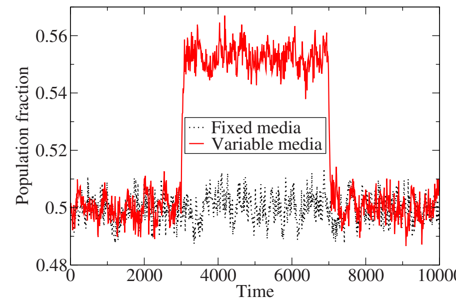

FIG. 5. (Color online) Plot of I_1 with respect to t using a fixed/variable media.

media opinion is very strong with respect to time. Figure 5 shows that the variation in I_1 is violent and that the result of the poll, which was previously balanced, is suddenly artificially moved in favor of party P_1.

From a sociological point of view, the model forecasts a growth of positive opinions concerning party P_1, induced by a very small change in X^2 toward X^1, without reinforcing the opinion of media M_1.

5. Unique media or two media with half strength to represent the media opinion.

We compare the behavior of the distribution function in two similar cases.

Situation 3. Three media act on the population, with the following characteristics:

$$X^1 = X^2 = (0.4,-0.4), \quad X^3 = (-0.4,0.4),$$

$$\alpha_1 = \alpha_2 = 0.1\beta, \quad \alpha_3 = 0.2\beta.$$

Situation 4. Two media act on the population, with the following characteristics:

$$X^1 = (0.4,-0.4), \quad X^2 = (-0.4,0.4), \quad \alpha_1 = \alpha_2 = 0.2\beta.$$

If I_1 had been plotted, we could have checked that the amount of people in favor of party P_1 oscillated around 0.5 in both situations, and no conclusion could have been drawn only from this information. The distribution functions are plotted in Fig. 6.

We note that media X^1 and X^2 of situation 3 have the same opinion as media X^1 of situation 4, with the sum of the strength of the former being equal to the strength of the latter. At a theoretical level, since the interaction with media is described by a linear term of the model, the results of situations 3 and 4 should be identical.

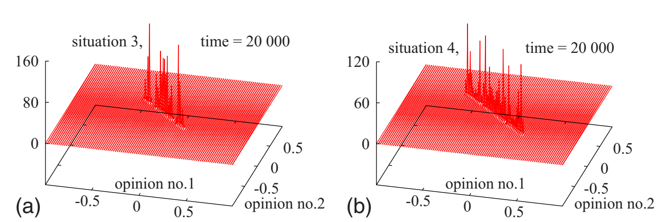

FIG. 6. (Color online) Distribution functions for situations (a) 3 and (b) 4, at $t=20\,000$.

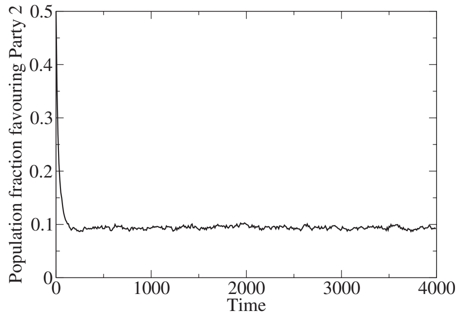

FIG. 7. Plot of I_2 with respect to t when the media rather favors party P_1.

We recover this feature in Fig. 6. The support of the two distribution functions is numerically identical and, moreover, their global shapes are similar. The differences in the shapes are originated by a numerical effect due to the treatment of the collisional part with a Bird method, based on a random routine.

6. One strong media against two weaker ones

We here investigate the situation of three media whose respective opinion vectors are

$$X^1 = (0.6, -0.6), \quad X^2 = (-0.6, 0.6), \quad X^3 = (-0.2, 0.2),$$

and their strengths

$$\alpha_1 = 0.2\beta, \quad \alpha_2 = \alpha_3 = 0.1\beta.$$

As expected, we can see in Fig. 7 that party P_2 is weakened by the media influence.

B. Three-party system

We now choose $p=3$; i.e., the phenomenon of opinion formation involves three political parties. Using a scalar opinion variable is not anymore an option.

Let

$$E_1 = \{x \in \Omega^3 | x_1 > \max(x_2, x_3)\},$$

$$E_2 = \{x \in \Omega^3 | x_2 > \max(x_1, x_3)\},$$

$$E_3 = \{x \in \Omega^3 | x_3 > \max(x_1, x_2)\}.$$

In the same way as in Sec. III A, we are interested in the three following integrals:

$$I_1 = \int\int\int_{E_1} f(x)dx, \quad I_2 = \int\int\int_{E_2} f(x)dx,$$

$$I_3 = \int\int\int_{E_3} f(x)dx,$$

which can be interpreted in terms of population fraction more likely to vote, in a proportional system, for parties P_1, P_2, and P_3, respectively. In all the tests, the population is uniformly distributed: $f^{\text{in}}(x) = 0.125$.

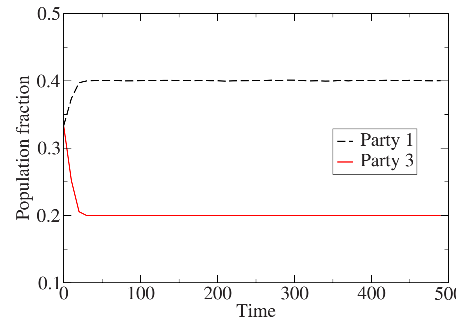

FIG. 8. (Color online) Plots of I_1 and I_3 with respect to t when party P_3 has no media support.

1. Medialess party

We investigate the situation where two of the three parties are supported by two media with the same strength and the last one has no mediatic support. More precisely, we have

$$X^1 = (0.4, -0.4, -0.4), \quad X^2 = (-0.4, 0.4, -0.4),$$

$$\alpha_1 = \alpha_2 = 0.1\beta.$$

Although it does not benefit from any mediatic help, supporters of party P_3 do not disappear, as one can check in Fig. 8. Of course, it is weakened with regard to the other parties, but still 20% of the population may eventually vote for it. The quantity I_2 is not plotted in Fig. 8; the two curves of I_1 and I_2 are almost superimposed: there is a total symmetry between the first and second variables.

2. Extremist media

We here consider a situation where party P_1 has a supporting media with a more asserted opinion; i.e., $X^1 = (0.9, -0.2, -0.2)$. The two other media support parties P_2 and P_3 with a more centered opinion; i.e., $X^2 = (-0.2, 0.3, -0.2)$ and $X^3 = (-0.2, -0.2, 0.3)$. The strength of each media is set to 0.1β.

In this situation, we can see in Fig. 9 that an extremist media does not really help the party which it supports: the moderate ones are far more efficient. Party P_1 has indeed the same result as if there were no media supporting it, as in Fig. 8.

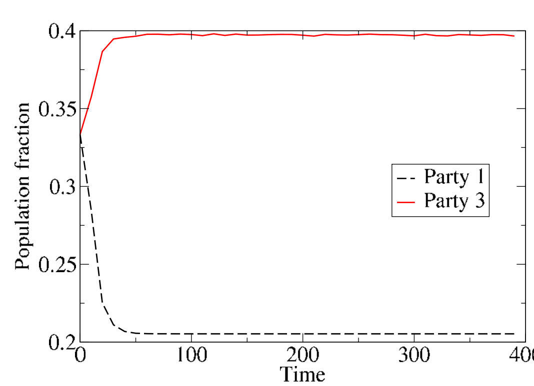

FIG. 9. (Color online) Plots of I_1 and I_3 with respect to t when party P_1 has an extremist support.

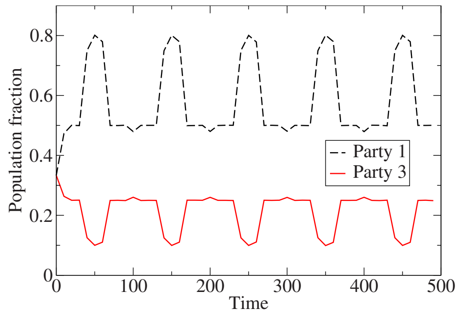

FIG. 10. (Color online) Plots of I_1 and I_3 with respect to t when party P_1 has a strong mediatic support.

3. Strong media with a variable opinion

Eventually, we study the behavior of our model in the case when there is one media stronger than the other ones and whose opinion varies with respect to t. More precisely, we set

$$X^1(t) = (0.3,-0.2,-0.2) + 0.2\cos(2\pi t/100)(-1,1,1),$$

$$X^2 = (-0.2,0.3,-0.2), \quad X^3 = (-0.2,-0.2,0.3),$$

and impose

$$\alpha_1 = 0.2\beta, \quad \alpha_2 = \alpha_3 = 0.1\beta.$$

Once again, the quantity I_2 is not plotted in Fig. 10 for symmetry reasons. The strength of media M_1, which is linked to party P_1, significantly increases the influence of this party. Moreover, the variations in X^1 induce some non-vanishing oscillations on I_1 and I_3, as seen in Fig. 10. When X_1^1 is close to its maximal value, around $t \equiv 50 (\mathrm{mod}\ 100)$, the proportion of the population which favors party P_1 is around 80% whereas it should be around 50%. On the contrary, when X_1^1 is close to its minimal value, party P_1 loses its absolute majority.

IV. MATHEMATICAL WELL POSEDNESS

We study here some mathematical properties of our problem [Eqs. (5) and (6)]. We first obtain an alternate weak formulation of our problem, and then deduce a theorem which asserts the existence and uniqueness of a solution to Eqs. (5) and (6).

A. Collision term

The form of the collisional integral given by Eq. (2) is not completely satisfactory for the gain term because of the intricate dependence of the argument of the test function on the variables x, x^*. We therefore consider the weak form of the gain term

$$\langle Q^+(f,f),\varphi\rangle = \int\int_{\Omega^{2p}} \beta(x,x^*)f(t,x)f(t,x^*)\varphi(x')dx^*dx. \tag{7}$$

In the same way as in [6], let us denote

$$D_\eta^i = \left\{ (x_i,x_i') \in \mathbb{R}\times\bar{\Omega} \,\middle|\, \frac{x_i'-1}{2} + \eta(x_i')\frac{x_i'+1}{2} \le x_i \le \frac{x_i'+1}{2} + \eta(x_i')\frac{x_i'-1}{2} \right\}$$

and

$$K_\eta(x,x') = \prod_{i=1}^{p} \frac{2}{1-\eta(x_i')}\chi_{D_\eta^i}(x_i,x_i'), \qquad \forall\, x,x' \in \bar{\Omega}^p,$$

where $\chi_{D_\eta^i}$ is the characteristic function of the set D_η^i. Since η is an admissible attraction function, it is clear that $D_\eta^i \subseteq \bar{\Omega}^2$. Note that, for a fixed η, K_η is obviously of class $L^\infty(\Omega^{2p})$.

We then perform the change of variables $x^* \mapsto x'$ in Eq. (7), for a fixed x. It is easy to see that

$$dx_i^* = \frac{2}{1-\eta(x_i)}dx_i' \quad \text{and} \quad x_i^* = \frac{2x_i'-x_i-\eta(x_i)x_i}{1-\eta(x_i)}.$$

Then, after permuting x and x', we obtain the following weak form of the gain term:

$$\langle Q^+(f,f),\varphi\rangle = \int\int_{\Omega^{2p}} \beta(x',y)K_\eta(x,x')f(t,y)f(t,x')\varphi(x)dx'dx, \tag{8}$$

where

$$y_i = \frac{2x_i - x_i' - \eta(x_i')x_i'}{1-\eta(x_i')}, \quad 1 \le i \le p.$$

Using Eq. (8) in Eq. (2), we obtain a new weak form of the collision operator, which becomes the definition of the collisional kernel in our model.

B. Linear term

We can obtain a similar result for the operator which models the interactions with a media. For a given j, we use again the notation Φ_i^j introduced in proposition 2.5. Let us then denote Ψ_i^j the inverse function of Φ_i^j and successively set, for any $z \in \bar{\Omega}^p$,

$$\Psi^j(z) = (\Psi_i^j(z_i))_{1\le i\le p}, \quad R^j(z) = \left(\prod_{i=1}^{p} (\Phi_i^j)'[\Psi_i^j(z_i)]\right)^{-1}.$$

Since Φ^j is a C^1 diffeomorphism, there exists $R>0$ such that the non-negative function R^j is upper bounded by R. We can now perform the change of variables $x \mapsto \tilde{x}$ in Eq. (4), permute x and $\tilde{x}$, and obtain a new weak form for the media action,

$$\langle L_j f,\varphi\rangle = \alpha_j \int_{\Omega^p} f(t,\Psi^j(x))R^j(x)\varphi(x)dx - \alpha_j \int_{\Omega^p} f(t,x)\varphi(x)dx.$$

C. Existence and uniqueness

We are now ready to prove the existence of weak solutions to our problem. The results are collected in the following theorem:

Theorem 4.1. *Let* f^{in} *a non-negative function of class* $L^1(\Omega^p)$. *Then, for all* $T>0$, *Eqs. (5) and (6) admit a unique non-negative solution* $f \in C^0([0,T];L^1(\Omega^p))$.

Proof. Let $T>0$. We consider the operator $\Theta: f \mapsto \Theta f$ defined on $C^0([0,T];L^1(\Omega^p))$, for $t \in [0,T]$ and $x \in \Omega^p$, by

$$\Theta f(t,x) = f(0,x) - \int_0^t \int_{\Omega^p} \beta(x,x') f(s,x) f(s,x') dx' ds + \int_0^t \int_{\Omega^p} \beta(x',y) K_\eta(x,x') f(s,y) f(s,x') dx' ds + \sum_{j=1}^m \alpha_j \int_0^t (f(s,\Psi^j(x)) R^j(x) - f(s,x)) ds,$$

where

$$y_i = \frac{2x_i - x_i' - \eta(x_i')x_i'}{1-\eta(x_i')}, \quad 1 \le i \le p.$$

It is a direct consequence from proposition 2.7 that $\Theta: C^0([0,T];L^1(\Omega^p)) \to C^0([0,T];L^1(\Omega^p))$.

The existence and uniqueness of a solution to Eqs. (5) and (6) follow if we can prove that Θ is a contraction in the functional space $C^0([0,T];L^1(\Omega^p))$. Indeed, Eq. (5) can be rewritten under a strong integral form as $\Theta f = f$. Let us hence consider u, $v \in C^0([0,T];L^1(\Omega^p))$ sharing the same initial condition f^{in}. We have

$$\|\Theta u - \Theta v\|_{L^\infty(0,T;L^1(\Omega^p))} \le T\left[2\|\beta K_\eta\|_{L^\infty(\Omega^{2p})}\|f^{\text{in}}\|_{L^1(\Omega)} + (1+R^p)\sum_{j=1}^m \alpha_j\right] \times \|u-v\|_{L^\infty(0,T;L^1(\Omega^p))}.$$

The quantity inside the square brackets in the previous inequality is a constant $A>0$ which only depends on the data. Hence, if we choose $T_0=(2A)^{-1}>0$, Θ is a contraction on $[0,T_0]$. Then there exists a unique $f \in L^\infty(0,T_0;L^1(\Omega^p))$ such that $\Theta f = f$. Thanks to the expression of Θf, it is then clear that, in fact, $f \in C^0([0,T_0];L^1(\Omega^p))$.

Moreover, since the mass is conserved by proposition 2.7, we can apply a bootstrap method by using as initial datum $f(T_0,x)$ and extend, if necessary, the time interval up to $[0,T]$. By induction, the existence and uniqueness of a solution to Eqs. (5) and (6) are proven. ■

ACKNOWLEDGMENTS

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